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Deduction of Heisenberg relations and Schrödinger equation through the structure of N-dimensional parameterized metric vector spaces

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Here, it is described how N-dimensional parameterized vector spaces, possessing an adapted real metric with the addition of some supplementary axioms, permit the deduction of Heisenberg relations and Schrödinger equation. This specific space structure, proposed as a container of the description of quantum objects, suggests, in this way, that both quantum mechanical cornerstones can be considered as trivial consequences of such a special vector space choice.

KEY WORDS: Heisenberg relations, Schrödinger equation, parameterized vector spaces, real valued metric, inward vector products

1. Introduction

Recent papers dealing with the role of Heisenberg relations [1] seems to conclude that uncertainty relationships constitute somehow a quantum theoretical cornerstone as well as an essential previous step towards setting up Schrödinger equation. Such an affirmation can be made in this way, as within the most recent paper [2], not only it is clearly shown that Heisenberg relations are deductible from classical considerations, under well defined statistical conditions, but Schrödinger equation can be derived from these relationships by means of further elegant theoretical deductions.

This situation has inspired a previous study [2], where it was shown Heisenberg relations could be also deduced by means of the definition of parameterized N-dimensional metric vector space structure. Such space structure, after some straightforward working definitions and a sequence of few ancillary axioms, presents the possibility of deducing the well-known uncertainty relationships, through the usual metric related constructs linking two vectors, as Gram matrix,

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cosine of the angle and Schwartz inequality. This previous space set up and the subsequent deduction opens the way to work out an extension of the Heisenberg relations and, then, permits deduce Schrödinger equation formalism. This powerful performance constitutes the main subject studied in this paper.

In order to proceed with this task, first a résumé of the previously described definitions will be given. It will be followed by a generalization of the results leading to Heisenberg relations, which will contain them as a particular case. From here, it will be presented the possibility to add to the deductive mechanisms the possible use of Euclidean distances and triangle inequality in order to obtain a similar relationship to Heisenberg uncertainty, as the one deduced employing the cosine of the angle subtended by two vectors definition or Schwartz inequality. Finally, Euclidean distances between N-dimensional parameterized metric space vectors, submitted to a simple variational procedure, can be shown as a natural path permitting to deduce a general equation structure, with the form of time independent Schrödinger secular equation.

2. Parameterized metric vector spaces

The sequence of definitions and the subsequent deductions needed to construct the Heisenberg relations and the Schrödinger equation, as mentioned in section 1, have to start with the statement consisting into that an *N-dimensional parameterized metric vector space* $V_N(\mathbf{C},t)$ can be defined by means of the equation

$$V_N(\mathbf{C},t) = \{x(t)|x(t) = \{x_I(t)\}\,(I = 1, N) \wedge t \in D \subseteq \mathbf{R}; \forall I : x_I(t) \in \mathbf{C}\}\,. \quad (1)$$

That is: every vector of $V_N(\mathbf{C},t)$ is supposedly composed by a N-dimensional ordered set of complex valued, continuous and differentiable functions of a real variable. Such variable, symbolized in turn by the parameter: t, is supposed to possess values contained within some domain D of the real field. It must be said now that this parametric monodimensional construction has been chosen for notation easiness sake. In this sense, following a previous study of such parameterized spaces [4], intended to deal with the description of general space–time frameworks, the space definition contained in equation (1) must be considered a simplified form of an, as complex as imaginable, general parameterized space structure, where the parameter t can be substituted by a properly defined parametric array of arbitrary dimensions. Such a construct, as the one chosen here, resembles the typical form of the Euclidean N-dimensional space in curvilinear coordinates [5].

2.1. Vector real inward product

The metric in $V_N(\mathbf{C},t)$ can be constructed, in addition, by means of setting a previous *real inward vector product* definition up. That is, a new *real* component vector can be arranged within $V_N(\mathbf{C}, t)$ simply, writing the algorithm

$$\forall x, y \in V_N(\mathbf{C}, t) \rightarrow \exists z = \{x : y\} = \tfrac{1}{2}(x^* * y + y^* * x) \in V_N(\mathbf{R}, t). \tag{2}$$

Where in equation (2) it has been used the *inward vector product* definition [6], which has been previously employed in several fields, mainly dealing with quantum mechanical applications [7]:

$$\forall a, b \in V_N(\mathbf{C}, t) \rightarrow c = a * b = \{c_I = a_I b_I\} \in V_N(\mathbf{C}, t).$$

It must be now realized that the real inward products, involving a unique vector, produce the elements of the associated *vector semispace*: $V_N(\mathbf{R}^+, t)$, [8] as it may be easily seen from equation (2):

$$\forall x \in V_N(\mathbf{C}, t) \rightarrow \exists r = \{x : x\} = x^* * x = \{r_I = |x_I|^2\} \equiv |x|^2 \in V_N(\mathbf{R}^+, t). \tag{3}$$

A vector semispace is defined over the positive definite real field and at the same time, the vector sum is provided with an additive abelian *semigroup* structure [9]. The vector semispaces characteristic features [8], as well as the most naturally appropriate metric definition [10] within such vector semispaces, have been recently published. The semispace vectors defined in equation (3) can be also connected with the usual structure of probability distributions and possess the power of being the basic composition from where a vector space can be made [11].

2.2. *Vector space real metric*

The definition of real inward vector products can produce in a natural way a convenient metric structure within N-dimensional parameterized vector spaces. In order to grasp this possibility, it is only necessary to properly propose a formal way to perform *real scalar products* within the complex parameterized space $V_N(\mathbf{C}, t)$. For that purpose, it can be defined, starting from the real inward product (2), the vector to scalar transformation: $V_N(\mathbf{C}, t) \rightarrow \mathbf{R}$, by means of employing the following algorithm:

$$\begin{aligned}
&\forall \{x : y\} \in V_N(\mathbf{R}, t) \rightarrow \\
&\qquad \langle\{x : y\}\rangle = \sum_I \int_D \{x : y\}_I dt = \int_D \left(\sum_I \{x : y\}_I dt\right) \\
&\qquad = \tfrac{1}{2} \sum_I \int_D \left(x_I(t)^* y_I(t) + y_I(t)^* x_I(t)\right) dt \in \mathbf{R}
\end{aligned}$$

thus, it is worthwhile to use a simpler notation to express the real scalar product. The following convention will be followed from now on:

$$\langle x : y\rangle \equiv \langle\{x : y\}\rangle.$$

When considering the $x = y$ case, then the real scalar product and the *Euclidean norm* coincide:

$$\langle\{x : x\}\rangle = \langle x : x\rangle = \sum_I \int_D |x_I|^2 dt = \int_D \left(\sum_I |x_I|^2\right) dt = \langle x|x\rangle \in \mathbf{R}^+.$$

3. Vector triads in parameterized metric vector spaces

3.1. Triads

In the space $V_N(\mathbf{C}, t)$ with a real metric, as defined in the section above, there in addition it can be chosen, for each non-zero vector element $x \in V(\mathbf{C}, t)$ a *triad:* $T(x)$. A triad is defined as a set of three vectors, composed by the triad *generator vector* coincident with the normalized vector:

$$|x\rangle = x(t) \in V_N(\mathbf{C}, t) \wedge \langle x : x\rangle = \langle x|x\rangle = 1$$

and a pair of *triad companion vectors*. The first triad companion is made by means of a vector, which can be written as:

$$|\theta x\rangle = \theta(t)x(t) \in V_N(\mathbf{C}, t), \tag{4}$$

or which can be constructed multiplying the triad generator by a well defined, continuous and differentiable real function $\theta(t)$ of the parameter t.

The second-triad companion vector is defined, in turn, as:

$$\frac{\partial}{\partial t}[x(t)] \equiv \frac{\partial x}{\partial t} \equiv \left|\frac{\partial x}{\partial t}\right\rangle \in V_N(\mathbf{C}, t) \tag{5}$$

and in this manner, it is considered made with the first derivative of the triad generator vector with respect the basic space parameter t. Such derivative can be defined in turn as:

$$\frac{\partial}{\partial t}[x\,(t)] = \left\{\frac{\partial}{\partial t}[x\,(t)]_I = \frac{\partial}{\partial t}[x_I\,(t)]\right\}.$$

In this manner, it can be admitted the definitions (4) and (5) could be used as additional axioms holding in the chosen parameterized metric vector spaces. Thus, a triad can be defined accordingly as:

$$T(x) = \left\{x; \theta x; \frac{\partial x}{\partial t}\right\} \equiv \left\{|x\rangle; \Big|\theta x\rangle; \left|\frac{\partial x}{\partial t}\right\rangle\right\} \subset V_N(\mathbf{C}, t).$$

Of course, both triad companions can be considered images of the complex parameterized space $V_N\,(\mathbf{C}, t)$ endomorphisms: $\theta\,(t)$ and $(\partial/\partial t)$.

3.2. Orthogonal relationships within a triad

Once a triad $T(x)$ is selected in $V_N(\mathbf{C}, t)$, then, the projector over the direction of the triad generator vector is easily constructed, as well as the projector over the associated orthogonal complement:

$$|x\rangle \to P_x = |x\rangle \langle x| \wedge P_{\tilde{x}} = I - P_x = I - |x\rangle \langle x| .$$

The triad companion vectors can be made orthogonal to the generator vector just by projecting them into the orthogonal complement subspace:

$$|\theta x\rangle \to |\theta \tilde{x}\rangle = P_{\tilde{x}} |\theta x\rangle \wedge \left|\frac{\partial x}{\partial t}\right\rangle \to \left|\frac{\partial \tilde{x}}{\partial t}\right\rangle = P_{\tilde{x}} \left|\frac{\partial x}{\partial t}\right\rangle,$$

which constitutes a procedure equivalent to the Gram–Schmidt orthonormalization algorithm, thus, after that, the projections are orthogonal to the triad generator vector:

$$\langle x \mid \theta \tilde{x}\rangle = 0 \wedge \left\langle x \;\middle|\; \frac{\partial \tilde{x}}{\partial t}\right\rangle = 0 \to \langle x : \theta \tilde{x}\rangle = \left\langle x : \frac{\partial \tilde{x}}{\partial t}\right\rangle = 0. \tag{6}$$

Nothing opposes to admit such a general feature can be considered active within each triad. Therefore, it can be supposed that the orthogonality relationships (6) hold within every triad. A similar consideration was made by Weyl [12], as a natural way to simplify the Heisenberg relations deduction. Thus, one can safely consider that the equivalent simplified notation can be employed within a triad from now on

$$|\theta x\rangle \equiv |\theta \tilde{x}\rangle \wedge \left|\frac{\partial x}{\partial t}\right\rangle \equiv \left|\frac{\partial \tilde{x}}{\partial t}\right\rangle.$$

3.3. Orthogonality relations and expectation values in a triad

The orthogonal relationships (6), can be interpreted as a convenient device, transferring into the triad companion vectors the additional property to possess a null expectation value, associated to the function $\theta(t)$ and the first derivative $(\partial/\partial t)$ operator. That is, the following equalities:

$$\langle \theta \rangle = \langle x : \theta x\rangle = \langle \theta \{x : x\}\rangle = \langle x \mid \theta x\rangle = 0 \tag{7}$$

hold and also at the same time the derivative operator corresponding ones:

$$\left\langle \frac{\partial}{\partial t}\right\rangle = \left\langle x : \frac{\partial x}{\partial t}\right\rangle = \left\langle \left\{x : \frac{\partial x}{\partial t}\right\}\right\rangle = \left\langle x \;\middle|\; \frac{\partial x}{\partial t}\right\rangle = 0. \tag{8}$$

A valuable consideration can be made at this moment, dealing with the parameter first derivative expectation values. In fact, equation (8), leading to the first parameter derivative expectation value of a given triad generator vector can be also alternatively written as:

$$\left\langle \frac{\partial}{\partial t} \right\rangle = \left\langle \frac{\partial}{\partial t} \{x : x\} \right\rangle = \frac{1}{2} \left\{ \left\langle \frac{\partial x^*}{\partial t} * x \right\rangle + \left\langle x^* * \frac{\partial x}{\partial t} \right\rangle \right\} = \left\langle x : \frac{\partial x}{\partial t} \right\rangle,$$

however, this property indicates the fact consisting in that the expectation value can be considered as the result of the following integral:

$$\left\langle \frac{\partial}{\partial t} \right\rangle = \left\langle \frac{\partial \rho}{\partial t} \right\rangle = \int_D \frac{\partial}{\partial t} [\rho(t)] \, \mathrm{d}t,$$

where the density function $\rho(t)$ is defined in the usual way, within the parameterized metric semispace $V_N\left(\mathbf{R}^+, t\right)$ elements:

$$\rho(t) = \{x : x\} = |x(t)|^2 .$$

4. Variances and covariance within triad companion vectors

Another important question, which turns to be extremely relevant for the progress of the subsequent theoretical development, corresponds to the form taken by *variances* associated to both triad companion vectors. Variances, for the operators involved in triad companion vectors definition, can be defined by means of the positive definite expectation values:

$$\left\langle \theta^2 \right\rangle = \langle \theta x : \theta x \rangle = \left\langle \theta^2 \{x : x\} \right\rangle = \langle \theta x \mid \theta x \rangle = \langle x | \theta^2 | x \rangle \in \mathbf{R}^+ \tag{9}$$

and

$$\left\langle \frac{\partial^2}{\partial t^2} \right\rangle = \left\langle \frac{\partial x}{\partial t} : \frac{\partial x}{\partial t} \right\rangle = \left\langle \frac{\partial x}{\partial t} \,\middle|\, \frac{\partial x}{\partial t} \right\rangle \in \mathbf{R}^+. \tag{10}$$

Equations (9) and (10) can be used in any case. However, it can be easily realized the convenience to make equation (10) able to be also represented by the equivalent axiomatic property:

$$\left\langle \frac{\partial x}{\partial t} : \frac{\partial x}{\partial t} \right\rangle = \left\langle \frac{\partial x}{\partial t} \,\middle|\, \frac{\partial x}{\partial t} \right\rangle \equiv \pm \left\langle x \,\middle|\, \frac{\partial^2}{\partial t^2} \,\middle|\, x \right\rangle, \tag{11}$$

which, when the minus sign holds or it is chosen for convenience, one can easily admit that the Green theorem [14] applies to the parameterized metric space vectors.

Variances, then, owing to the nullity of the expectation values, as expressed in equations (7) and (8), become coincident with the expectation values (10) and (11), or:

$$\mathrm{var}\,(\theta) = \left\langle \theta^2 \right\rangle \wedge \mathrm{var}\left(\frac{\partial}{\partial t}\right) = \left\langle \frac{\partial^2}{\partial t^2} \right\rangle. \tag{12}$$

At the same time, it is easy to see that the scalar product of the triad companion vectors, can be interpreted as the covariance between the associated operators, or:

$$\left\langle \theta x : \frac{\partial x}{\partial t} \right\rangle = \mathrm{cov}\left(\theta; \frac{\partial}{\partial t}\right). \tag{13}$$

5. Gram matrix, angle subtended and Schwartz inequality associated to triad companion vectors

Knowing the previous definitions and properties of a triad $T\,(x)$ in $V_N(\mathbf{C}, t)$, it is easy to write the relevant part of a *Gram matrix* for a triad, which is made by a (2×2) matrix, composed by the real scalar products involving the triad companion vectors:

$$\mathbf{G} = \begin{pmatrix} \langle \theta x \mid \theta x \rangle & \left\langle \theta x : \frac{\partial x}{\partial t} \right\rangle \\ \left\langle \theta x : \frac{\partial x}{\partial t} \right\rangle & \left\langle \frac{\partial x}{\partial t} \mid \frac{\partial x}{\partial t} \right\rangle \end{pmatrix},$$

this is so, as the triad generator is orthogonal to the companion vectors, and therefore the first row and column of the (3×3) triad Gram matrix are coincident with the unit matrix ones, and consequently both become irrelevant for the following development. The *Gramian*, simply the determinant of the Gram matrix, is easily developed to fulfill the inequality:

$$Det\,|\mathbf{G}| = \langle \theta x \mid \theta x \rangle \left\langle \frac{\partial x}{\partial t} \,\middle|\, \frac{\partial x}{\partial t} \right\rangle - \left\langle \theta x : \frac{\partial x}{\partial t} \right\rangle^2 > 0, \tag{14}$$

provided the companion vectors are admittedly considered to be linearly independent and, in this manner, the Gram matrix, being positive definite, leads to the relationship (14). Equation (14) represents the same algorithm as to use the definition of the *cosine of the angle* subtended by the involved companion vectors, as it can be written:

$$\cos^2(\alpha) = \left\langle \theta x : \frac{\partial x}{\partial t} \right\rangle^2 \left(\langle \theta x \mid \theta x \rangle \left\langle \frac{\partial x}{\partial t} \,\middle|\, \frac{\partial x}{\partial t} \right\rangle \right)^{-1} \in [0, 1].$$

In any case, both expressions lead to the so-called *Schwartz inequality*, which can be written as:

$$\langle \theta x \mid \theta x \rangle \left\langle \frac{\partial x}{\partial t} \,\middle|\, \frac{\partial x}{\partial t} \right\rangle \geqslant \left\langle \theta x : \frac{\partial x}{\partial t} \right\rangle^2 , \tag{15}$$

owing to the fact that some situations, where the companion vectors could be linearly dependent, that is: scalar multiples, can be also taken into account.

6. General form of Heisenberg relations

The mathematical structure of the N-dimensional parameterized metric vector spaces is already perfectly set in order to deduce Heisenberg relations. First, it has to be considered equation (15) and then using equations (9)–(12) afterwards, in order to obtain the inequality:

$$\operatorname{var}(\theta) \operatorname{var}\left(\frac{\partial}{\partial t}\right) \geqslant \left\langle \theta x : \frac{\partial x}{\partial t} \right\rangle^2 . \tag{16}$$

Also, owing to the covariance definition (13), involving the triad companion vectors, then equation (16) above can be alternatively written as:

$$\operatorname{var}(\theta) \operatorname{var}\left(\frac{\partial}{\partial t}\right) \geqslant \operatorname{cov}\left(\theta; \frac{\partial}{\partial t}\right)^2 .$$

6.1. Preliminary considerations

Now, the usual treatment can be followed in order to handle the right-hand side of equation (16). Subsequently, just to obtain equivalent elements as those found in the literature, see for example references [13, 15], it can be written:

$$\left\langle \frac{\partial}{\partial t} \{x : \theta x\} \right\rangle = \left\langle \frac{\partial x}{\partial t} : \theta x \right\rangle + \left\langle x : \frac{\partial \theta}{\partial t} x \right\rangle + \left\langle x : \theta \frac{\partial x}{\partial t} \right\rangle . \tag{17}$$

First, one can consider as a natural axiom the situation where the right hand side integral, involving the companion vectors, can be made a constant:

$$\left\langle \frac{\partial}{\partial t} \{x : \theta x\} \right\rangle = \omega,$$

which, when $\theta(t) \equiv t$, can be supposed null [13]. Also, it is easy to find that:

$$\left\langle x : \theta \frac{\partial x}{\partial t} \right\rangle = \left\langle \theta x : \frac{\partial x}{\partial t} \right\rangle = \left\langle \frac{\partial x}{\partial t} : \theta x \right\rangle$$

and moreover, it can be taken in consideration the middle integral in equation (17) is expressible as a norm of the triad generator vector, weighted by the derivative $(\partial\theta/\partial t)$. Furthermore, such a norm coincides with the unity, when the triad companion operator is chosen as: $\theta(t) \equiv t$. Therefore, using these ideas, after providing this weighted norm expression with a specific value:

$$\left\langle x : \frac{\partial\theta}{\partial t} x \right\rangle = \eta,$$

it can be also written:

$$\omega = \eta + 2\left\langle x : \theta \frac{\partial x}{\partial t} \right\rangle \rightarrow \left\langle x : \theta \frac{\partial x}{\partial t} \right\rangle = \frac{1}{2}(\omega - \eta) = \frac{1}{2}\lambda,$$

which in the case $\theta(t) \equiv t$, will be transformed simply in $\lambda = -1$. In the present discussion, it is sufficient to consider as an axiom that $\lambda \neq 0$ holds. Of course, owing to equation (13) it can be also written:

$$\operatorname{cov}\left(\theta; \frac{\partial}{\partial t}\right) = \frac{1}{2}\lambda.$$

This constant triad companions covariance result can be simply admitted because of the constant nature of integral (17), but also precludes, in fact, even a better situation: it corresponds to an alternative property with respect Heisenberg relations. Such a characteristic was discussed in a previous work [2], coinciding with an earlier appreciation of Bohm [15] dealing on the importance of covariance in quantum mechanical theory.

6.2. Heisenberg relations

Thus, substituting this result in equation (16), one obtains:

$$\operatorname{var}(\theta)\operatorname{var}\left(\frac{\partial}{\partial t}\right) \geqslant \frac{1}{4}\lambda^2,$$

which constitutes the Heisenberg relations for a given triad definition.

Uncertainties being defined as the standard deviations or the square roots of variances:

$$\Delta(\theta) = \sqrt{\operatorname{var}(\theta)} \wedge \Delta\left(\frac{\partial}{\partial t}\right) = \sqrt{\operatorname{var}\left(\frac{\partial}{\partial t}\right)},$$

produce the more classical looking literature expression [16] form:

$$\Delta(\theta)\,\Delta\left(\frac{\partial}{\partial t}\right) \geqslant \frac{1}{2}\lambda,$$

coinciding with the usual Heisenberg relations when $\theta(t) \equiv t$ and thus: $|\lambda| = 1$.

7. Triad Euclidean distances

After the previous discussion, one can wonder about the Euclidean distances can be used for some similar purposes, as the angle subtended by the triad companion vectors, in order to find out Heisenberg relations or equivalent expressions. This section is devoted to discuss such kind of possibilities.

7.1. Euclidean distances

Suppose a triad $T(x)$ in $V(\mathbf{C}, t)$, the squared Euclidean distance of the triad generator vector to the first triad companion can be written:

$$D^2(x; \theta x) = \langle x \mid x \rangle + \langle \theta x \mid \theta x \rangle - 2 \langle x : \theta x \rangle,$$

however, due to the normalization of the triad generator plus the orthogonality relation (7), it can be finally written:

$$D^2(x; \theta x) = 1 + \langle \theta x \mid \theta x \rangle = 1 + \operatorname{var}(\theta).$$

Similar considerations allow writing for the squared Euclidian distance between generator and second companion triad vectors:

$$D^2\left(x; \frac{\partial x}{\partial t}\right) = 1 + \operatorname{var}\left(\frac{\partial}{\partial t}\right).$$

Finally considering the couple of triad companion vectors it can easily be written:

$$D^2\left(\theta x; \frac{\partial x}{\partial t}\right) = \operatorname{var}(\theta) + \operatorname{var}\left(\frac{\partial}{\partial t}\right) - 2\left\langle \theta x : \frac{\partial x}{\partial t}\right\rangle,$$

which, owing to the previous considerations about the value of the last integral right hand term, also representing the triad companions covariance, can be finally written as:

$$D^2\left(\theta x; \frac{\partial x}{\partial t}\right) = \operatorname{var}(\theta) + \operatorname{var}\left(\frac{\partial}{\partial t}\right) - \lambda. \tag{18}$$

7.2. Heisenberg relations from Euclidean distances

After considering the above expressed squared distances in terms of the variances of the operators entering the triad companions, the triangular inequality can be invoked to obtain, after squaring both sides:

$$D^2(x; \theta x) + D^2\left(x; \frac{\partial x}{\partial t}\right) + 2D(x; \theta x)\, D\left(x; \frac{\partial x}{\partial t}\right) \geqslant D^2\left(\theta x; \frac{\partial x}{\partial t}\right),$$

substituting the squared distance values previously obtained, eliminating afterwards equal terms, appearing on both inequality sides and finally rearranging the constants, the new inequality is found:

$$D\left(x;\theta x\right)D\left(x;\frac{\partial x}{\partial t}\right)\geqslant-\left(1+\frac{1}{2}\lambda\right)\equiv-\sigma,$$

which, after squaring both sides again, produces the expression:

$$\left(1+\mathrm{var}\left(\theta\right)\right)\left(1+\mathrm{var}\left(\frac{\partial}{\partial t}\right)\right)\geqslant\sigma^2,$$

which also can be considered as constituting an alternative form of Heisenberg relations.

8. Schrödinger equation deduced from Euclidean distances between triad companion vectors

8.1. *Euclidean triad companion distance as quadratic error*

Coming back to the Euclidean distance between the triad companion vectors, as deduced and expressed in equation (18), there can be alternatively written:

$$D^2\left(\theta x;\frac{\partial x}{\partial t}\right)=\langle\theta x\mid\theta x\rangle+\left\langle\frac{\partial x}{\partial t}\,\middle|\,\frac{\partial x}{\partial t}\right\rangle-\lambda=\left\langle\left(\theta x-\frac{\partial x}{\partial t}\right):\left(\theta x-\frac{\partial x}{\partial t}\right)\right\rangle\equiv\mathrm{e}^{(2)},$$

where $\mathrm{e}^{(2)}$ can be associated to the quadratic error between the triad companion vectors. Of course, one can expect, owing to the nature of the companion vectors, that:

$$\mathrm{e}^{(2)}\geqslant 0,$$

where the zero value has to be taken as meaning the coincidence of both triad companion elements. Nothing opposes that this null situation can be taken as a restrictive axiom, accepting in this way the fact that it could never happen within a triad.

8.2. *Variation of the quadratic error*

This state of affairs developed up to now, also allows admitting the possible presence, in the parameterized metric vector space, of some set of variational parameters, which can be considered as embedded into the vector component functions.

The procedure of optimal variation of the quadratic error with respect these parameters, first provides the augmented Lagrange function $\Lambda(x)$, as a consequence of submitting the optimal search to the restriction of the triad generator vector unit norm:

$$\Lambda(x) = \langle x | \theta^2 | x \rangle - \left\langle x \left| \frac{\partial^2}{\partial t^2} \right| x \right\rangle - \lambda - \varepsilon \left(\langle x \mid x \rangle - 1 \right),$$

where the minus sign in the second-order derivative integral has been chosen for convenience and ε is an undetermined Lagrange multiplier. Variation of the augmented function, remembering that λ has to be considered a constant, provides:

$$\delta\Lambda(x) = \langle \delta x | \theta^2 | x \rangle - \left\langle \delta x \left| \frac{\partial^2}{\partial t^2} \right| x \right\rangle - \varepsilon \langle \delta x \mid x \rangle = 0,$$

which leads to the Euler equation:

$$\theta^2 | x \rangle - \frac{\partial^2}{\partial t^2} | x \rangle - \varepsilon | x \rangle = 0. \tag{19}$$

Equation (19) is equivalent to a classical time independent Schrödinger equation. It can be rewritten trivially, by using the secular equation formalism:

$$\left(-\frac{\partial^2}{\partial t^2} + \theta^2 \right) | x \rangle = \varepsilon | x \rangle . \tag{20}$$

It is also straightforward to see that equation (20) transforms into the Schrödinger harmonic oscillator equation when choosing: $\theta(t) = t$. At the same time, introducing some potential function by means of considering: $\theta(t) = \sqrt{V(t)}$, the following equation:

$$\left(-\frac{\partial^2}{\partial t^2} + V(t) \right) | x \rangle = \varepsilon | x \rangle$$

is obtained, which lacks of the quantum one half factor in the second derivative. However, this constant can be considered just as a scale factor, affecting the potential and the Lagrange multiplier. Obviously enough, the Lagrange multiplier can be associated to the quantum system energy.

9. Conclusions

The properly defined N-dimensional parameterized metric vector spaces permit to set up Heisenberg's uncertainty relations first and, then, Schrödinger equation afterwards. After realizing this, it may be concluded that, not only these fundamental quantum mechanical ideas are easily deducible from the space mathematical structure, containing basic quantum mechanical object descriptors appropriately set, but it can be easily admitted that both, Heisenberg relations and Schrödinger equation, have an unavoidable connection with the natural characteristics of such spaces.

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